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A Theory of the Eddy Current Equivalent Winding and its Application to the Closing of Non-Delayed Telephone Relays (A study of Telephone Relays-4)

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A THEORY OF THE EDDY CURRENT EQUIVALENT WINDING AND ITS APPLICATION TO THE CLOSING OF NON-DELAYED TELEPHONE RELAYS

A STUDY OF
TELEPHONE RELAYS
4

BY

STIG EKELÖF

GÖTEBORG 1961
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A Theory of the Eddy Current Equivalent Winding and its Application to the Closing of Non-Delayed Telephone Relays

A Study of Telephone Relays (4)

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In this paper the influence of the eddy currents on a telephone relay of the electromagnetic type on closing is investigated. The same relay as in Parts 1, 2, and 3 of the study is considered, i.e., an "L-armature" relay with a round core and a rectangular yoke, where the magnetic leakage is taken care of by the simplified field picture introduced in Part 1. The relay is assumed to have eddy currents in core and yoke. The armature is kept fixed and the magnetic reluctances are supposed to be constant.

The concept of an equivalent winding, representing the eddy current paths in a magnetic circuit, is subjected to a general theoretical treatment. It is shown that an equivalent winding should give useful results on closing but cannot be employed on breaking.

For the relay under consideration the equivalent winding parameters on closing are deduced. Numerical results are reported. The importance of the magnetic skin effect is pointed out.

CHAPTER 1

Introduction and summary

In the present Part 4 of "A Study of Telephone Relays" the same type of relay is considered as in the earlier Parts 1, 2, and 3[2,3,4], *i.e.*, a relay of the L-armature type with a round core and a rectangular yoke. As before, the armature is kept fixed, the magnetic reluctances are assumed to be constant, and the magnetic leakage is represented by a leakage flux, running perpendicularly between core and yoke.

The present paper and Part 3 are closely connected inasmuch as both papers discuss the same aspect of relay theory, *viz.*, the influence of the eddy currents arising in the metallic parts of the magnetic circuit. As in Part 3 we assume eddy currents in both core and yoke. For some general points of view reference is made to Part 3, Chapter 1.

Employing the concept of operational reluctances, we could in Part 3 derive without difficulty general operational expressions for the magnetic core flux on closing or breaking, and for the magnetizing primary current on closing. On *breaking*—the problem studied in detail in Part 3—it was then possible to deduce from the operator of the magnetic flux (only the armature end flux was considered) two different expressions for the corresponding time function: Heaviside's expansion theorem gave us an exponential function series, suitable for large values of the time t; a series expansion of the operator in powers of $p^{-1/4}$ gave us a series in powers of $t^{1/4}$, suitable for small t. These two solutions were used for studying quantitatively an actual relay. The results yielded by the study were in excellent agreement with actual oscillograph records of the flux decay.

In this paper we turn to a detailed study of the problem of *closing*, trying to obtain suitable expressions for the primary current and the magnetic flux. Unfortunately, in this case the method of Part 3 seems to be excluded on account of numerical complications (as to the expansion theorem solution see the remark following eq. (119) in Part 3; as to the power series solution, given by eq. (124), this solution can only be used for extremely small values of t, and further terms of the series are difficult to obtain, as they require a better approximation of the quantity $\bar{\eta}^2$ than the one given by (72)).

A new approach is necessary. In the following it will be shown that on closing sufficiently accurate solutions can be obtained by *the method of replacing the eddy current paths by an equivalent winding*.

The idea of an equivalent winding was mentioned already in Part 3, Chapter 1. It was pointed out that some earlier writers[7,8,9] have made use of this idea, interpreting more or less explicitly the equivalent winding as a physical wire winding, wound around the relay core (*cf.* also a paper by H. HESS and E. WEBER[6]).

The results of Part 3 strongly supported the assertion that the breaking problem cannot be studied by the aid of a single equivalent winding. This procedure leads to an exponential decrease of the flux-time curve, an outcome which is very far from the actual course of events. Other and more suitable methods could, however, be found.

For the closing problem, on the other hand, where—as explained above—these methods break down, the concept of an equivalent winding is useful. One of the aims of the present paper is to develop a satisfactory general theory of the equivalent winding.

Chapter 2 deals with the basic principles of the equivalent winding, and gives general expressions for its parameter values, as depending upon the eddy current distribution. In *Chapter 3* these expressions are used for an approximate calculation of the equivalent winding parameters of a relay on closing.

In *Chapter 4* the results are tested on a case which can also be exactly solved, *viz.*, a relay without magnetic leakage and with eddy currents in the core only. The comparison with the exact solution turns out satisfactorily. In *Chapter 5* the equivalent winding is then used for a quantitative study of the closing of a relay *with* magnetic leakage and with eddy currents in core *and* yoke. In order to facilitate the computations, a list is finally given of the *Principal symbols and formulæ*.

CHAPTER 2

General theory of the eddy current equivalent winding

This chapter gives a general theory of the equivalent winding of the eddy currents in a magnetic circuit. The theory is presented as an application to our telephone relay. It is quite general, however, and can in principle be applied to any magnetic circuit.

We consider a relay according to *Fig. 1* on the next page (*cf.* Part 1, Fig. 4; Part 2, Fig. 9; Part 3, Fig. 2) with a cylindrical core and a rectangular yoke. The core and the yoke are assumed to be the only electrically conducting parts of the magnetic circuit. A primary magnetizing wire winding with N_1 turns is evenly distributed over the whole length of the core. The relay has a magnetic leakage flux $\varphi_\lambda(x, t)$ per unit length, running perpendicularly between core and yoke. Hence, the magnitude of the main flux $\Phi(x, t)$ passing through core and yoke varies with the distance x from the heel piece end.

If the current i_1 in the wire winding undergoes variations, we obtain through electromagnetic induction secondary or eddy currents in the electrically conducting parts of the magnetic circuit. According to Lenz' law the eddy currents give rise to opposing magnetic fluxes, which tend to slow down the rate of change of the total flux through a core section and to speed up the rate of change of the primary current.

We shall show the following. If the geometric pattern of the eddy current distribution remains constant in time, it is possible to replace the eddy currents as to their influence on the

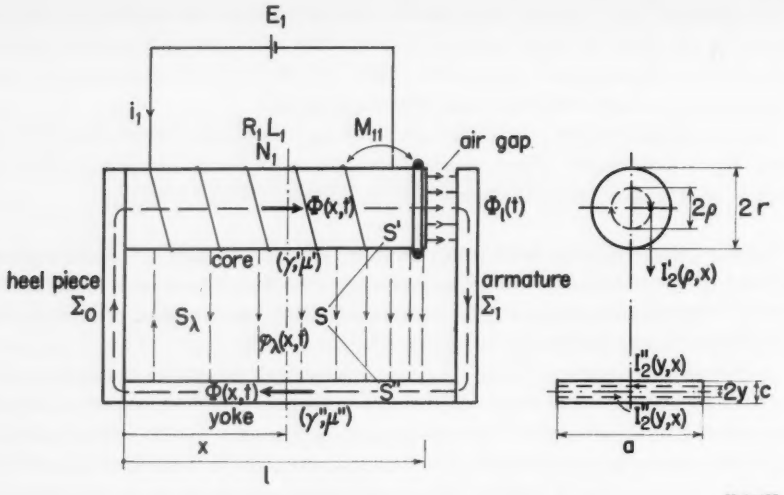

Fig. 1. Idealized relay with magnetic leakage.

primary current and on the magnetic fluxes by an equivalent winding with constant parameter values (resistance R_2, inductance L_2, mutual inductances M and M_{21}), carrying a current i_2 = the entire circulating secondary current. General expressions for the parameter values will be given.

Hence, under the condition mentioned, we can for computational purposes replace a relay with eddy currents by a relay without eddy currents but with a delaying wire winding. Thus we can employ the theory of such a relay, given in Part 2. It should be stressed that the equivalent winding is a fictitious winding, which we can use formally in our calculations even if we may not be able to realize it physically.

Units and symbols

As in the earlier parts of this study the rationalized MKS-system of units is used throughout.

In this part we denote

quantities pertaining to the core by the superscript ′.
" " " " yoke " " " ″.
" " " " primary (magnetizing) winding by the subscript 1.
" " " " eddy current system and the secondary (equivalent) winding by the subscript 2.

As far as possible the symbols of Parts 1 to 3 are employed. See Fig. 1 and the list of symbols at the end of the paper.

Definitions of the secondary current and of the parameters of the equivalent winding

We assume that the eddy currents in core and yoke run in planes perpendicular to the direction of x, and that their paths are symmetrical with respect to the core axis and to the symmetry planes of the yoke. This means (see Fig. 1):

the eddy current paths in the core are concentric circles; the current density I_2' is a function of the radius ϱ of the path and of its distance x from the heel piece end,

$$I_2' = I_2'(\varrho, x) \tag{1 a}$$

the eddy current paths in the yoke are parallel straight lines, with a direction parallel to the yoke surfaces; the current density I_2'' is a function of the distance y from the centre plane and of the distance x,

$$I_2'' = I_2''(y, x) \tag{1 b}$$

Of course, I_2' and I_2'' are in general also functions of the time t.

Hence, we have a circulating current in the core,

$$i_2' = \int_{x=0}^{l} \int_{\varrho=0}^{r} I_2'(\varrho, x)\, d\varrho\, dx \tag{2 a}$$

and in the yoke

$$i_2'' = \int_{x=0}^{l} \int_{y=0}^{c/2} I_2''(y, x)\, dy\, dx \tag{2 b}$$

We now define the current in the equivalent winding, the *secondary current* i_2, by the condition that it should be equal to the entire circulating current,

$$i_2 = i_2' + i_2'' \tag{3}$$

We define the resistance of the equivalent winding, the *secondary resistance* R_2, by the condition that the heat developed when the current i_2 passes through R_2 be equal to the entire eddy current heat,

$$R_2 i_2^2 = \int_{\text{core}} \frac{I_2'^2}{\gamma'}\, d\tau' + \int_{\text{yoke}} \frac{I_2''^2}{\gamma''}\, d\tau'' \tag{4}$$

We define the inductance of the equivalent winding, the *secondary inductance* L_2, by the condition that the magnetic energy arising when the current i_2 passes through L_2 be equal to the entire magnetic energy of the eddy currents. Hence, introducing the magnetic vector potentials $\mathbf{A}_2'$ and $\mathbf{A}_2''$ of the eddy current system, we shall have[1]

$$\frac{1}{2} L_2 i_2^2 = \int_{\text{core}} \frac{1}{2} \mathbf{I}_2' \cdot \mathbf{A}_2'\, d\tau' + \int_{\text{yoke}} \frac{1}{2} \mathbf{I}_2'' \cdot \mathbf{A}_2''\, d\tau'' \tag{5}$$

We finally define the *mutual inductance* M between the primary wire winding and the equivalent winding by the condition that the mutual magnetic energy of the primary current i_1 through the wire winding and the secondary current i_2 through the equivalent winding be equal to the mutual magnetic energy of the primary current and the eddy current system.

Now, as is well known, the mutual energy can be expressed in two different ways, leading to two formally different expressions for M.

If we use the vector potentials $\mathbf{A}_1'$, $\mathbf{A}_1''$ of the primary current magnetic field and the eddy current vectors $\mathbf{I}_2'$, $\mathbf{I}_2''$, we have

$$M i_1 i_2 = \int_{\text{core}} \mathbf{I}_2' \cdot \mathbf{A}_1' \, d\tau' + \int_{\text{yoke}} \mathbf{I}_2'' \cdot \mathbf{A}_1'' \, d\tau'' \tag{6 a}$$

Here the integrals should be taken over the regions which carry eddy currents, *i.e.*, over the volumes of the core and the yoke.

If, instead, we use the vector potential $\mathbf{A}_2$ of the eddy current magnetic field and the vector $\mathbf{I}_1$ of the primary current, we have

$$M i_1 i_2 = \int_{\substack{\text{primary} \\ \text{winding}}} \mathbf{I}_1 \cdot \mathbf{A}_2 \, d\tau \tag{6 b}$$

Here the integral should be taken over the region which carries the primary current, *i.e.*, over the copper volume of the primary wire winding.

Our use of the vector potentials in expressing the magnetic energies brings great advantages to our treatment inasmuch as we then only have to integrate over the current-carrying regions, while if we use expressions of the type $\int 1/2 \, \mathbf{B} \cdot \mathbf{H} \, d\tau$, the integration must be extended to the entire volume of the magnetic field.

Integration of (6 b)

It is easy to solve formally the integral in (6b). It yields the expression for M given in (9), which will prove to be of great importance to us.

In carrying out the integration, we neglect the pitch of the winding, thus assuming that the current paths are circles, running perpendicularly to the core axis. We also assume that the primary winding has a very small thickness ε (see *Fig. 2*). This last assumption simplifies the calculations but is otherwise not essential.

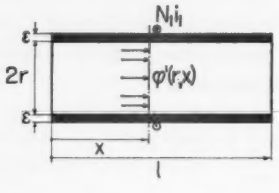

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Fig. 2. To the integration of (6 b).

Introducing in (6b)

$$I_1 = \frac{N_1 i_1}{\varepsilon l}; \quad d\tau = 2\pi r \varepsilon \, dx$$

we obtain

$$M i_2 = \int_{x=0}^{l} 2\pi r A_{2\alpha} \frac{N_1}{l} \, dx \tag{7a}$$

where $A_{2\alpha}$ denotes the component of $\mathbf{A}_2$ in the direction of $\mathbf{I}_1$, that is, in a direction tangential to the circumference of the core section.

Now, an integral $\oint A_s ds$ being equal to the magnetic flux linked with the closed path of integration, we have

$$2\pi r A_{2\alpha} = \varphi'(r, x) \tag{8}$$

where $\varphi'(r, x)$ = the total eddy current core flux at the distance x from the heel piece end. Thus, from (7a):

$$M i_2 = \int_{x=0}^{l} \frac{N_1 dx}{l} \varphi'(r, x) \tag{7b}$$

Evidently, the last integral is equal to the total eddy current magnetic flux Φ_{21} linked with the primary winding. Hence,

$$M i_2 = \Phi_{21} \tag{9}$$

Thus, this formula, well-known from the theory of filiform circuits, is valid for our equivalent winding in the sense that the flux Mi_2 which the current in the equivalent winding sends through the primary winding, is equal to the eddy current flux Φ_{21}.

Validity of the equivalent winding

Evidently, the values of the quantities R_2, L_2, and M, which follow from (4), (5), and (6a) or (6b), are determined by the eddy current distribution at the moment in question. If our equivalent winding shall be practically useful, these values must be constant, independent of time. It is easy to see that, in accordance with our earlier prediction, R_2, L_2, *and* M *are constant if the geometric distribution of the eddy currents is independent of time, i.e.*, if we can write

$$I_2'(\varrho, x) = f(\varrho, x)\psi(t); \quad I_2''(y, x) = g(y, x)\psi(t)$$

with the same function $\psi(t)$ in both expressions.

If R_2, L_2, and M are constant, we can prove by energy considerations in conjunction with the relation (9) that the equivalent winding gives us the correct values of the currents i_1 and i_2.

We start with the power relation

$$E_1 i_1 = (R_1 i_1^2 + R_2 i_2^2) + \frac{dW}{dt} \tag{10}$$

expressing that the battery power $E_1 i_1$ is used up partly as Joulean heat $R_1 i_1^2 + R_2 i_2^2$, partly for increasing the magnetic energy

$$W = \frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2 \tag{11}$$

Employing (11), we can write (10) as

$$i_1 \left(R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} - E_1 \right) + i_2 \left(R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \right) = 0 \tag{12}$$

Now, for the primary circuit we have the relation

$$R_1 i_1 + L_1 \frac{di_1}{dt} = E_1 + \left(- \frac{d\Phi_{21}}{dt} \right) \tag{13}$$

or, making use of (9):

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} - E_1 = 0 \tag{14}$$

The expression within the first pair of brackets in (12) being thus = 0, it follows that the expression within the second pair of brackets must also be = 0, *i.e.*,

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = 0 \tag{15}$$

Evidently, (14) and (15) are the equations governing the system formed by the primary circuit and the short-circuited equivalent winding. Hence, *this coupled system yields correct values of the primary current i_1 and the total circulating eddy current i_2.*

The armature end flux

We are above all interested in the total armature end flux Φ_l. Following the procedure introduced in Part 2, Chapter 3, we express Φ_l by the aid of two mutual inductances, *viz.*,

M_{11} = the mutual inductance between the primary winding and a fictitious winding of one turn at the armature end,

M_{21} = the mutual inductance between the equivalent winding and the fictitious armature end winding.

Φ_l is produced partly by the primary current i_1, giving the contribution $M_{11}i_1$, and partly by the eddy currents. Making use of the notation $\varphi'(r, x)$ of eq. (8), we can express the contribution from the eddy currents as $\varphi'(r, l)$. We write this

$$\varphi'(r, l) = M_{21}i_2 \tag{16}$$

and take (16) as the definition of M_{21}. It is evident that M_{21} also is constant if the geometric distribution of the eddy currents is independent of time.

We thus obtain for the armature end flux

$$\Phi_l = M_{11}i_1 + M_{21}i_2 \tag{17}$$

i.e., (*cf.* Part 2, eq. (34)) the same formula as for a relay with a delaying wire winding.

A physical picture of the equivalent winding

Although it is hardly possible to materialize the equivalent winding, we can without difficulty form a physical picture of it. For simplicity we assume eddy currents in the core only. The general eddy current path then consists of a circle with the radius ϱ around the core axis.

Fig. 3 shows two perpendicular sections of the core, one through the axis and one at right angles to it.

We make use of the following symbols, dropping for the moment the superscript $'$ and the index 2.

V = core volume (volume element $d\tau$).

S = surface, on one side of core axis, of a longitudinal core section (hatched in Fig. 3; surface element da).

I = current density.

$U = 2\pi\varrho E$ = voltage around an eddy current path ($E = I/\gamma$ = electric field strength).

$\varphi = 2\pi\varrho A$ = total eddy current flux, linked with an eddy current path (A = vector potential of eddy current magnetic field).

$\Phi = 2\pi\varrho A_1$ = primary current flux, linked with an eddy current path (A_1 = vector potential of primary current magnetic field).

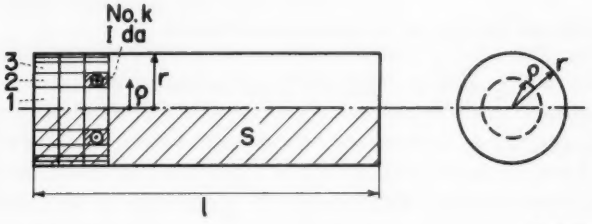

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Fig. 3. The physical realization of the equivalent winding.

According to the definition of i_2 we have

$$i_2 = \int_S I\,da \tag{18}$$

Further, eq. (4), defining R_2, gives us

$$R_2 i_2^2 = \int_V \frac{I^2}{\gamma}\,d\tau = \int_S I \cdot E \cdot 2\pi\varrho\,da$$

or

$$R_2 i_2 = \int_S U\,\frac{I\,da}{i_2} \tag{19}$$

In the same manner we obtain from (5) and (6a):

$$L_2 i_2 = \int_S \varphi\,\frac{I\,da}{i_2} \tag{20}$$

$$M i_1 = \int_S \Phi\,\frac{I\,da}{i_2} \tag{21}$$

Now let us assume (see Fig. 3) that we divide the surface S into a large but finite number n of elements da, chosen so that the same current $I\,da$ passes through all of them. We then obtain n circular turns with radii ϱ between 0 and r, all carrying the same current $I\,da = i_2/n$.

We number the turns from 1 to n and denote by U_k, φ_k, and Φ_k the values of U, φ, and Φ, pertaining to turn No. k. Then

$$U_k = -\frac{d\varphi_k}{dt} - \frac{d\Phi_k}{dt} \tag{22}$$

and

$$\sum_1^n U_k = -\frac{d}{dt}\sum_1^n \varphi_k - \frac{d}{dt}\sum_1^n \Phi_k \tag{23}$$

Eqs. (19), (20), and (21) give us the approximate relations

$$R_2 i_2 = \frac{1}{n}\sum_1^n U_k;\quad L_2 i_2 = \frac{1}{n}\sum_1^n \varphi_k;\quad M i_1 = \frac{1}{n}\sum_1^n \Phi_k \tag{24 a, b, c}$$

Hence, from (23):

$$R_2 i_2 + L_2\,\frac{di_2}{dt} + M\,\frac{di_1}{dt} = 0 \tag{25 a}$$

i.e., the relation (15) already deduced.

Since the n turns all carry the same current i_2/n, they can be connected in series. We then obtain a coil with a certain resistance R_2^*, a certain inductance L_2^*, and a certain mutual inductance M^*. This coil will have a voltage between its ends of $\sum_1^n U_k$ and it will link the magnetic fluxes $\sum_1^n \varphi_k$ and $\sum_1^n \Phi_k$. Hence

$$R_2^* \frac{i_2}{n} = \sum_1^n U_k; \quad L_2^* \frac{i_2}{n} = \sum_1^n \varphi_k; \quad M^* i_1 = \sum_1^n \Phi_k \qquad (26\,\text{a, b, c})$$

Comparing (24 a, b, c) and (26 a, b, c) we find

$$R_2^* = n^2 R_2; \quad L_2^* = n^2 L_2; \quad M^* = nM \qquad (27\,\text{a, b, c})$$

From (23) follows

$$R_2^* \frac{i_2}{n} + L_2^* \frac{d}{dt} \frac{i_2}{n} + M^* \frac{di_1}{dt} = 0 \qquad (25\,\text{b})$$

We can now summarize our investigation as follows:

Pursuing the eddy current paths, we divide the entire core into a great number of turns n, each turn carrying the same current i_2/n. We insulate the turns from each other by very thin insulating layers, cut them open, connect them in series, and short-circuit the coil thus obtained. This coil will have a resistance R_2^*, an inductance L_2^*, and a mutual inductance M^*. For the short-circuited coil the relation (25 b) is valid.

We could use this coil as our equivalent winding but for one serious disadvantage: in the limit $n \to \infty$ we would obtain a coil with infinitely large parameters, carrying an infinitely small current.

Therefore, it is more convenient to employ an entirely formal equivalent winding with the finite parameters

$$R_2 = \lim_{n\to\infty} \frac{R_2^*}{n^2}; \quad L_2 = \lim_{n\to\infty} \frac{L_2^*}{n^2}; \quad M = \lim_{n\to\infty} \frac{M^*}{n} \qquad (28\,\text{a, b, c})$$

and carrying the finite current i_2. If we short-circuit this winding, eq. (25 a) = (15) holds. This equation differs from (25 b) only by the factor $1/n$.

The necessity of the condition that the geometric distribution of the eddy currents should be independent of time now becomes clear. This condition ensures that the n turns of our coil always carry equal currents, so that they can be connected in series, forming a fixed coil with *constant* parameters. Unfortunately this case is an exception; normally the eddy current pattern varies from one moment to the next, so that the configuration of the coil would have to change continuously. In other words, in most cases the equivalent winding cannot be used, because its parameters would vary with time. This is the reason why *the equivalent winding is of no value in a study of the relay breaking problem.*

On the other hand, the equivalent winding yields useful approximate results in such cases where the magnetic field produced by the eddy currents is small as compared with the main field produced by the current in a magnetizing wire winding. This latter field can then be considered as the only cause of the eddy currents, which can therefore be calculated from a homogeneously distributed magnetic flux varying with time. The parameters R_2, L_2, and M will then be constant in time and easy to compute.

If the eddy current magnetic field, although not small, has but little influence on the primary current and on the total magnetic flux, the same method can be reckoned upon to yield acceptable values of these two quantities. We shall make use of this when treating in the next chapter the relay closing problem. In this case we cannot expect very good results in the first moments where the eddy currents cause a quite large rate of increase of the primary current.

CHAPTER 3

Parameters of the eddy current equivalent winding of a relay on closing

Distribution along the core of the primary current magnetic flux

A constant primary current i_1 gives rise to a magnetic flux Φ_1 in core and yoke, the magnitude of which, on account of the magnetic leakage flux φ_λ, varies with the distance x from the heel piece end. As shown in Part 1, the function $\Phi_1 = \Phi_1(x)$ can with sufficient accuracy be approximated with a parabola (Part 1, eqs. (28) and (20 a)).

We introduce the quantities

$$a = \frac{\Sigma_0}{S_\lambda}; \quad b = \frac{\Sigma_1}{S_\lambda} \tag{29 a, b}$$

$$\eta = \sqrt{\frac{S}{S_\lambda}}; \quad \eta' = \sqrt{\frac{S'}{S_\lambda}}; \quad \eta'' = \sqrt{\frac{S''}{S_\lambda}} \tag{30 a, b, c}$$

$$g^2 = \eta^2 \frac{ab\,\dfrac{\sinh\eta}{\eta} + a\cosh\eta + b}{\eta^2\,\dfrac{\sinh\eta}{\eta} + b(\cosh\eta - 1)} \tag{31 a}$$

Putting

$$\frac{x}{l} = u$$

we can write the results of Part 1:

$$\Phi_1(x) = \Phi_1(0)\,\lambda(u) \tag{32a}$$

with

$$\lambda(u) = 1 + au - \frac{1}{2}\,g^2u^2 \tag{32b}$$

and

$$\Phi_1(0) = \frac{N_1 i_1/S_\lambda}{g^2 + \eta^2} \tag{32c}$$

Generally $\eta^2 \ll 1$. Then

$$g^2 \approx \frac{a + b + ab + \frac{1}{2}\,\eta^2 a\left(1 + \frac{1}{3}\,b\right)}{1 + \frac{1}{2}\,b + \frac{1}{6}\,\eta^2\left(1 + \frac{1}{4}\,b\right)} \tag{31b}$$

In our subsequent deductions the parabolic function $\lambda(u)$ gives rise to the following easily calculated average values:

$$m(u) = \int_0^u \lambda(u)\,du = u(1 + au/2 - g^2u^2/6) \tag{33a}$$

$$n(u) = 2\int_0^u m(u)\,du = u^2(1 + au/3 - g^2u^2/12) \tag{33b}$$

$$m_1 = m(1) \qquad = 1 + a/2 - g^2/6 \tag{34a}$$

$$n_1 = n(1) \qquad = 1 + a/3 - g^2/12 \tag{34b}$$

$$p_1 = 3\int_0^1 n(u)\,du \qquad = 1 + a/4 - g^2/20 \tag{34c}$$

$$q_1 = \int_0^1 \lambda(u)^2\,du \qquad = 1 + a - (g^2 - a^2)/3 - ag^2/4 + g^4/20 \tag{34d}$$

$$r_1 = \int_0^1 (1 + au)\,\lambda(u)\,du = 1 + a - (g^2 - 2a^2)/6 - ag^2/8 \tag{34e}$$

$$t_1 = 3\int_0^1 \lambda(u)\,n(u)\,du \qquad = 1 + a - (7g^2/4 - a^2)/5 - ag^2/8 + g^4/56 \tag{34f}$$

The eddy currents

In calculating the eddy currents, we proceed as stated in Chapter 2, *i.e.*, neglecting their own magnetic flux, we assume that they wholly result from the primary flux caused by the current i_1 through the magnetizing winding. This flux is uniformly distributed across core and yoke and varies parabolically along them; it changes with time in the same way as i_1.

Hence, it can be written

$$\Phi_1(x, t) = \lambda(u) f(t) \tag{35}$$

where the function $f(t)$ need not be specified.

1) The eddy currents in the core.

Applying Faraday's law to a circular path of radius ϱ, we obtain

$$2\pi\varrho \frac{I_2'(\varrho, x)}{\gamma'} = -\frac{d}{dt}\left\{\frac{\varrho^2}{r^2}\Phi_1(x, t)\right\} \tag{36a}$$

or, introducing $\Phi_1(x, t)$ from (35) and putting $-df(t)/dt = g(t)$:

$$I_2'(\varrho, x) = \frac{\gamma'}{2\pi r}\frac{\varrho}{r}\lambda(u) g(t) \tag{37a}$$

For the entire circulating current in the core we obtain

$$i_2' = \int_{x=0}^{l}\int_{\varrho=0}^{r} I_2' \, d\varrho \, dx = \frac{m_1 l}{4}\frac{\gamma'}{\pi} g(t) \tag{38a}$$

2) The eddy currents in the yoke.

Here we apply Faraday's law to a rectangular path of length a and width $2y$:

$$2a \frac{I_2''(y, x)}{\gamma''} = -\frac{d}{dt}\left\{\frac{2y}{c}\Phi_1(x, t)\right\} \tag{36b}$$

Thus,

$$I_2''(y, x) = \frac{\gamma''}{2a}\frac{y}{c/2}\lambda(u) g(t) \tag{37b}$$

and the entire circulating current

$$i_2'' = \int_{x=0}^{l}\int_{y=0}^{c/2} I_2'' \, dy \, dx = \frac{m_1 l}{4}\frac{c/2}{a}\gamma'' g(t) \tag{38b}$$

3) The eddy current densities, expressed in the entire circulating current.

For the entire circulating current—by definition also the current in the equivalent winding —we obtain

$$i_2 = i_2' + i_2'' = \frac{m_1 l}{4}\left(\frac{\gamma'}{\pi} + \gamma''\frac{c/2}{a}\right) g(t) \tag{39}$$

We introduce the two quantities

$$\delta' = \frac{\gamma'/\pi}{\gamma'/\pi + \gamma'' c/2a}; \quad \delta'' = \frac{\gamma'' c/2a}{\gamma'/\pi + \gamma'' c/2a} \tag{40 a, b}$$

δ' and δ'' show how the current i_2 is divided between core and yoke:

$$i_2' = \delta' i_2; \quad i_2'' = \delta'' i_2; \quad \delta' + \delta'' = 1 \tag{41 a, b, c}$$

Employing now (37 a, b), (39), and (40 a, b), we obtain the following final expressions for the eddy current densities:

$$I_2'(\varrho, x) = \frac{2\delta'}{m_1 l r} \frac{\varrho}{r} \lambda(u) \cdot i_2 \tag{42 a}$$

$$I_2''(y, x) = \frac{2\delta''}{m_1 l c/2} \frac{y}{c/2} \lambda(u) \cdot i_2 \tag{42 b}$$

The eddy current magnetic field

The calculation of L_2, M, and M_{21} requires that we know the magnetic field of the eddy currents. We need (see *Fig. 4*) the following two quantities:

$\varphi'(\varrho, x) = \int_0^{\varrho} \mu' H'(\varrho, x)\, 2\pi\varrho\, d\varrho =$ the eddy current magnetic flux in the core, linked with a circular eddy current path with the radius ϱ, situated at the distance x from the heel piece end.

$\varphi''(y, x) = \int_0^{y} \mu'' H''(y, x)\, 2a\, dy =$ the eddy current magnetic flux in the yoke, linked with a rectangular eddy current path with the length a and the width $2y$, situated at the distance x from the heel piece end.

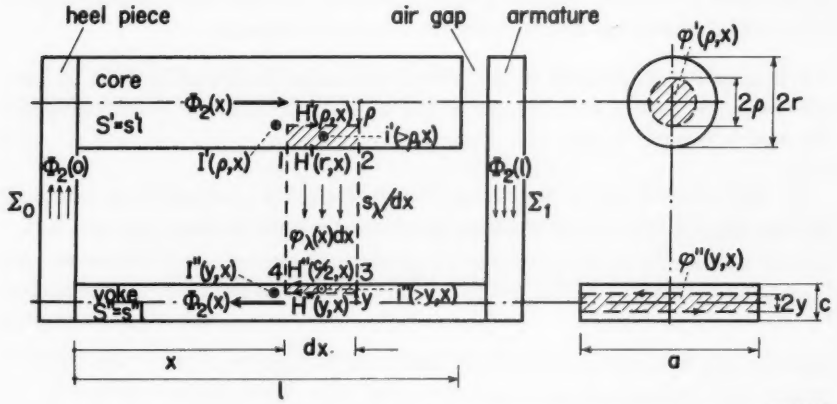

Fig. 4. To the calculation of the eddy current magnetic fluxes.

We also introduce

$\Phi_2(x)$ = the total eddy current flux in core or yoke at the distance x from the heel piece end.

Evidently

$$\Phi_2(x) = \varphi'(r,\ x) = \varphi''(c/2,\ x) \tag{43}$$

In Part 1, Chapter 2, and again in Part 2, Chapter 3, are given the relations which govern the magnetic fluxes Φ and φ_λ produced by a constant current i_1 through a magnetizing wire winding covering the whole length of the core. These relations consist of two differential equations (Part 2, eqs. (62) and (63)) and two end conditions (Part 2, eqs. (64) and (65)).

In our present case, where the fluxes are produced by the eddy currents, the first differential equation, expressing the flux continuity, can be taken over unchanged:

$$\varphi_\lambda = -\frac{d\Phi_2}{dx} \tag{44}$$

The same holds good for the end conditions:

$$\Sigma_0 \Phi_2(0) + s_\lambda \varphi_\lambda(0) = 0; \quad \Sigma_1 \Phi_2(l) - s_\lambda \varphi_\lambda(l) = 0 \tag{45 a, b}$$

In Part 2 we used the symbols Φ and φ_λ. In (44) and (45 a, b) we have replaced Φ by Φ_2 but for simplicity we have retained φ_λ although in the present case this flux also derives from the eddy currents.

There remains the second differential equation,

$$s\Phi + s_\lambda \frac{d\varphi_\lambda}{dx} = \frac{N_1 i_1}{l} \tag{46}$$

obtained as an application of Ampère's law. This equation implies a uniformly distributed flux Φ, caused by a current i_1 circulating on the core surface. In our present case this condition is not fulfilled, whence the equation cannot be taken over unchanged.

In fact, the flux Φ_2 is caused by eddy currents circulating *within* core and yoke, and the flux distribution across a section of core or yoke is strongly non-uniform. We shall show that (46) must be replaced by (51).

We apply Ampère's law to the rectangular path 1234 in Fig. 4, situated in the interspace between core and yoke. No current being linked with the path, we obtain

$$s_\lambda \varphi_\lambda(x + dx) - s_\lambda \varphi_\lambda(x) + \{H'(r,\ x) + H''(c/2,\ x)\}\, dx = 0$$

or

$$H'(r,\ x) + H''(c/2,\ x) = -s_\lambda \frac{d\varphi_\lambda}{dx} \tag{47}$$

where $H'(r,\ x)$ and $H''(c/2,\ x)$ are the magnetic intensities at the surfaces of core and yoke.

We next apply Ampère's law to the two rectangular paths enclosing the hatched areas within the core and the yoke, and obtain

$$H'(\varrho, x) - H'(r, x) = i'(>\varrho, x) \tag{48a}$$

$$H''(y, x) - H''(c/2, x) = i''(>y, x) \tag{48b}$$

where $i'(>\varrho, x)$ = the total circulating current per unit core length outside a cylinder with radius ϱ.

$i''(>y, x)$ = the total circulating current per unit yoke length outside a rectangular strip with the transverse dimensions a and $2y$.

Using (42 a, b), we immediately calculate

$$i'(>\varrho, x) = \int_{\varrho}^{r} I_2'(\varrho, x)\, d\varrho = \frac{\delta'}{m_1 l}\left\{1 - \left(\frac{\varrho}{r}\right)^2\right\} \lambda(u) \cdot i_2 \tag{49a}$$

$$i''(>y, x) = \int_{y}^{c/2} I_2''(y, x)\, dy = \frac{\delta''}{m_1 l}\left\{1 - \left(\frac{y}{c/2}\right)^2\right\} \lambda(u) \cdot i_2 \tag{49b}$$

We now turn to the two expressions for Φ_2 given in (43):

$$\Phi_2(x) = \int_0^r \mu' H'(\varrho, x)\, 2\pi\varrho\, d\varrho = \int_0^{c/2} \mu'' H''(y, x)\, 2a\, dy$$

Introducing here $H'(\varrho, x)$ and $H''(y, x)$ from (48 a, b), and making use of (49 a, b), we find

$$H'(r, x) = s'\Phi_2(x) - \frac{\delta'}{2m_1 l}\, \lambda(u) \cdot i_2 \tag{50a}$$

$$H''(c/2, x) = s''\Phi_2(x) - \frac{2\delta''}{3m_1 l}\, \lambda(u) \cdot i_2 \tag{50b}$$

Introducing these expressions in (47) and observing the relations $s' + s'' = s$, $\delta' + \delta'' = 1$, we obtain the following equation, replacing (46):

$$s\Phi_2 + s_\lambda \frac{d\varphi_\lambda}{dx} = \frac{1 + \delta''/3}{2m_1 l}\, \lambda(u) \cdot i_2 \tag{51}$$

Our next task is to find $\Phi_2 = \Phi_2(x)$ from (44), (45 a, b), and (51). Comparing (51) and (46), we see that for the purpose of computing Φ_2 we can still assume a current on the core surface. This current, however, is not independent of x as in Parts 1 and 2 but varies parabolically along the core.

In principle, an exact integration of the system (44, 51) offers no difficulties. Still, we prefer an approximate method, which, while being sufficiently accurate, involves much simpler calculations.

The term $s\Phi_2$ in (51) represents the magnetic voltage drop per unit length of core + yoke, a quantity which is generally quite small. We can therefore, without substantial loss of accuracy, replace it by an approximate expression, which is known to us. With $s\Phi_2$ known, an integration immediately gives us φ_λ from (51), and by a subsequent integration we compute Φ_2 (a better value than the one used in $s\Phi_2$!) from (44). Finally, the end conditions (45 a, b) give us the two integration constants.

As our approximate value of $s\Phi_2$ we choose the value, obtained by replacing the parabolically distributed current in (51) with a constant current, giving the same total mmf. We are thus brought back to the relation (46) so that Φ_2 follows at once from the results of Parts 1 and 2.

We denote the constant current per unit core length by i_2^*. Hence, equalizing the mmf's:

$$i_2^* l = \int_0^l \frac{1+\delta''/3}{2m_1 l}\,\lambda(u)\,i_2\,dx = \frac{1+\delta''/3}{2}\,i_2$$

i.e.

$$i_2^* = \frac{1+\delta''/3}{2l}\,i_2 \tag{52}$$

Substituting now in (32 c) $\Phi_2(0)$ for $\Phi_1(0)$ and $i_2^* l$ for $N_1 i_1$, we find

$$S\Phi_2(0) = i_2^* l\,\frac{\eta^2}{g^2+\eta^2}$$

and hence, observing that $S = ls$:

$$s\Phi_2(x) = s\Phi_2(0)\,\lambda(u) = \frac{1+\delta''/3}{2l}\,\frac{\eta^2}{g^2+\eta^2}\,\lambda(u)\cdot i_2 \tag{53}$$

Using this expression for $s\Phi_2$, we obtain from (51):

$$s_\lambda\,\frac{d\varphi_\lambda}{dx} = \lambda(u)\cdot i_2^{**} \tag{54}$$

with

$$i_2^{**} = \frac{1+\delta''/3}{2m_1 l}\left(1 - \frac{m_1\eta^2}{g^2+\eta^2}\right) i_2 \tag{55}$$

Integrating (54), while observing (33 a), we find

$$\varphi_\lambda(x) = \varphi_\lambda(0) + \frac{i_2^{**} l}{s_\lambda}\,m(u) \tag{56}$$

We now integrate (44), making use of (56), (33 b), and the relation $s_\lambda = lS_\lambda$. It follows that

$$\Phi_2(x) = \Phi_2(0) - x\varphi_\lambda(0) - \frac{i_2^{**} l}{2S_\lambda}\,n(u) \tag{57}$$

The unknown constants $\Phi_2(0)$ and $\varphi_\lambda(0)$ are determined by (45 a, b). One finds that (57) can be written

$$\Phi_2(x) = \Phi_2(0)\,(1 + au) - \frac{i_2^{**} l}{2S_\lambda}\, n(u) \tag{58}$$

with

$$\Phi_2(0) = \beta\,\frac{i_2^{**} l}{S_\lambda}; \quad \beta = \frac{m_1 + \frac{1}{2}\, bn_1}{a + b + ab} \tag{59 a, b}$$

Knowing Φ_2, we can now determine the magnetic field intensities $H'(\varrho, x)$ and $H''(y, x)$. The eqs. (48 a, b) together with (49 a, b) and (50 a, b) give us

$$H'(\varrho, x) = s'\,\Phi_2(x) + \frac{\delta'}{2m_1 l}\left\{1 - 2\left(\frac{\varrho}{r}\right)^2\right\}\lambda(u)\cdot i_2 \tag{60 a}$$

$$H''(y, x) = s''\,\Phi_2(x) + \frac{\delta''}{3m_1 l}\left\{1 - 3\left(\frac{y}{c/2}\right)^2\right\}\lambda(u)\cdot i_2 \tag{60 b}$$

Introducing $\Phi_2(x)$ from (58), and paying attention to (55) and (59 a, b), we finally obtain

$$H'(\varrho, x) = \frac{1}{2m_1 l}\left\{\eta'^2\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \delta'\left[1 - 2\left(\frac{\varrho}{r}\right)^2\right]\right\}\lambda(u)\cdot i_2 \tag{61 a}$$

$$H''(y, x) = \frac{1}{2m_1 l}\left\{\eta''^2\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \frac{2\delta''}{3}\left[1 - 3\left(\frac{y}{c/2}\right)^2\right]\right\}\lambda(u)\cdot i_2 \tag{61 b}$$

From the expressions (61 a, b) we compute without difficulty the fluxes $\varphi'(\varrho, x)$ and $\varphi''(y, x)$. The result is

$$\varphi'(\varrho, x) = \left\{\frac{1 + \delta''/3}{2m_1 S_\lambda}\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left(\frac{\varrho}{r}\right)^2\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \frac{\delta'}{2m_1 S'}\left(\frac{\varrho}{r}\right)^2\left[1 - \left(\frac{\varrho}{r}\right)^2\right]\lambda(u)\right\}\cdot i_2 \tag{62 a}$$

$$\varphi''(y, x) = \left\{\frac{1 + \delta''/3}{2m_1 S_\lambda}\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\frac{y}{c/2}\left[\beta\,(1 + au) - \frac{1}{2}\, n(u)\right] + \right.$$
$$\left. + \frac{\delta''}{3m_1 S''}\,\frac{y}{c/2}\left[1 - \left(\frac{y}{c/2}\right)^2\right]\lambda(u)\right\}\cdot i_2 \tag{62 b}$$

As a check we compute $\Phi_2(x)$ from both expressions:

$$\Phi_2(x) = \varphi'(r, x) = \varphi''(c/2, x) =$$

$$= \frac{1 + \delta''/3}{2m_1 S_\lambda} \left(1 - \frac{m_1 \eta^2}{g^2 + \eta^2}\right) \left[\beta (1 + au) - \frac{1}{2} n(u)\right] \cdot i_2 \tag{63}$$

and observe that the same value of $\Phi_2(x)$ follows from (58), combined with (55) and (59 a, b).

The resistance of the equivalent winding

From eq. (4) follows

$$R_2 = \int_{x=0}^{l} \int_{\varrho=0}^{r} \frac{(I_2'/i_2)^2}{\gamma'} \, 2\pi\varrho \, d\varrho \, dx + \int_{x=0}^{l} \int_{y=-c/2}^{+c/2} \frac{(I_2''/i_2)^2}{\gamma''} \, a \, dy \, dx \tag{64}$$

Taking I_2', I_2'' from (42 a, b) and performing the integrations, we easily find

$$R_2 = \frac{q_1}{m_1^2 l} \left(\frac{2\pi}{\gamma'} \delta'^2 + \frac{16a/3c}{\gamma''} \delta''^2\right) \tag{65}$$

The inductance of the equivalent winding

According to eq. (5) we have

$$L_2 = L_2' + L_2'' \tag{66}$$

with

$$L_2' = \frac{1}{i_2^2} \int_{\text{core}} \mathbf{I}_2' \cdot \mathbf{A}_2' \, d\tau'; \quad L_2'' = \frac{1}{i_2^2} \int_{\text{yoke}} \mathbf{I}_2'' \cdot \mathbf{A}_2'' \, d\tau'' \tag{67 a, b}$$

Reasoning as when deducing (20), we obtain

$$L_2' = \int_{x=0}^{l} \int_{\varrho=0}^{r} \frac{I_2'}{i_2} \frac{\varphi'}{i_2} \, d\varrho \, dx; \quad L_2'' = \int_{x=0}^{l} \int_{y=0}^{c/2} \frac{I_2''}{i_2} \frac{\varphi''}{i_2} \, dy \, dx \tag{68 a, b}$$

With the aid of (42 a, b) and (62 a, b) this gives us

$$L_2' = \frac{\delta'}{12 m_1^2 S_\lambda} \left\{\left(1 + \frac{\delta''}{3}\right) \left(1 - \frac{m_1 \eta^2}{g^2 + \eta^2}\right) \left(3\beta r_1 - \frac{1}{2} t_1\right) + q_1 \frac{\delta'}{\eta'^2}\right\} \tag{69 a}$$

$$L_2'' = \frac{\delta''}{12 m_1^2 S_\lambda} \left\{\frac{4}{3} \left(1 + \frac{\delta''}{3}\right) \left(1 - \frac{m_1 \eta^2}{g^2 + \eta^2}\right) \left(3\beta r_1 - \frac{1}{2} t_1\right) + \frac{16 q_1 \delta''}{15 \eta''^2}\right\} \tag{69 b}$$

and, if we make use of $\delta' + \delta'' = 1$:

$$L_2 = \frac{1}{12 m_1^2 S_\lambda} \left\{\left(1 + \frac{\delta''}{3}\right)^2 \left(1 - \frac{m_1 \eta^2}{g^2 + \eta^2}\right) \left(3\beta r_1 - \frac{1}{2} t_1\right) + q_1 \left(\frac{\delta'^2}{\eta'^2} + \frac{16 \delta''^2}{15 \eta''^2}\right)\right\} \tag{70}$$

The mutual inductance between the equivalent winding and the primary winding

For the mutual inductance M we derive two different expressions, one from (6 a) and the other from (6 b).

Taking first (6 a), we proceed as when calculating L_2 and find

$$M = M' + M'' \tag{71}$$

with

$$M' = \int_{x=0}^{l} \int_{\varrho=0}^{r} \frac{I_2'}{i_2} \frac{\Phi_1'}{i_1} \, d\varrho \, dx; \quad M'' = \int_{x=0}^{l} \int_{y=0}^{c/2} \frac{I_2''}{i_2} \frac{\Phi_1''}{i_1} \, dy \, dx \tag{72 a, b}$$

Here

$\Phi_1' = \Phi_1'(\varrho, x)$ = the primary current magnetic flux in the core, linked with a circular eddy current path of radius ϱ situated at the distance x from the heel piece end.

$\Phi_1'' = \Phi_1''(y, x)$ = the primary current magnetic flux in the yoke, linked with a rectangular eddy current path of length a and width $2y$ situated at the distance x from the heel piece end.

Introducing the total primary current flux

$$\Phi_1(x) = \Phi_1'(r, x) = \Phi_1''(c/2, x)$$

we have

$$\Phi_1'(\varrho, x) = \left(\frac{\varrho}{r}\right)^2 \Phi_1(x); \quad \Phi_1''(y, x) = \frac{y}{c/2} \Phi_1(x) \tag{73 a, b}$$

with $\Phi_1(x)$ according to (32 a, b, c).

We now integrate (72 a, b). The result is

$$M' = \frac{N_1}{2m_1 S_\lambda} \frac{\delta'}{g^2 + \eta^2} q_1; \quad M'' = \frac{N_1}{2m_1 S_\lambda} \frac{4\delta''/3}{g^2 + \eta^2} q_1 \tag{74 a, b}$$

and

$$M = \frac{N_1}{2m_1 S_\lambda} \left(1 + \frac{\delta''}{3}\right) \frac{q_1}{g^2 + \eta^2} \tag{75}$$

We pass on to (6 b). This relation has already been transformed to (7 b), which can be written

$$M = \frac{N_1}{l} \int_{x=0}^{l} \frac{\varphi'(r, x)}{i_2} \, dx \tag{76}$$

Introducing $\varphi'(r, x)$ from (63), we find

$$M = \frac{N_1}{2m_1 S_\lambda} \left(1 + \frac{\delta''}{3}\right) \left(1 - \frac{m_1 \eta^2}{g^2 + \eta^2}\right) \left[\beta\left(1 + \frac{a}{2}\right) - \frac{1}{6} p_1\right] \tag{77}$$

Generally, (75) and (77) lead to somewhat different values of M. This is due to the fact that the expression (63) for the eddy current flux $\varphi'(r, x)$ depends on the approximate value of $s\Phi_2$ which we used in (51). For normal values of $S = sl$, however, (75) and (77) yield very nearly the same numerical result. This confirms that our approximation of $s\Phi_2$ was permissible.

It is not difficult to see from (51) that in two cases, for $S = 0$ and for $S_\lambda = \infty$, our method of solution is exact. By comparing (75) and (77) one also finds that in both of these cases the two expressions for M coincide.

The mutual inductance between the equivalent winding and a fictitious winding of one turn at the armature end

Eq. (16) immediately gives us

$$M_{21} = \frac{\varphi'(r, l)}{i_2} \tag{78}$$

where $\varphi'(r, l)$ follows from (63). It ensues

$$M_{21} = \frac{1}{2m_1 S_\lambda}\left(1 + \frac{\delta''}{3}\right)\left(1 - \frac{m_1\eta^2}{g^2 + \eta^2}\right)\left[\beta\,(1 + a) - \frac{1}{2}\,n_1\right] \tag{79}$$

CHAPTER 4

The closing of relays without magnetic leakage and with eddy currents in the core only

In this chapter we study a simplified case, accessible to an exact treatment. It allows us to judge the usefulness of the equivalent winding solution by comparing it with the exact solution.

The exact solution

a) *Expansion theorem solution for large t.*

As mentioned in Part 3, a paper by I. HERLITZ[5], published in 1920, implicitly contains the solution of the eddy current problem for a relay with eddy currents in the core only, and without magnetic leakage. Making use of Heaviside's expansion theorem, Herlitz could express his quantities by means of exponential function series valid for not too small values of

time t. His expression for the magnetic flux, written with the symbols of our telephone relay problem, is presented in Part 3, eq. (29).

We give below the expansion theorem solutions for the following four quantities:

$i_1 = i_1(t)$ = the primary current.

$\Phi = \Phi(t)$ = the magnetic flux.

$H = H(t, \varrho)$ = the magnetic intensity in the core.

$I = I(t, \varrho)$ = the eddy current density in the core.

(ϱ = distance from the core axis.)

The subsequent notations are used:

$$\tau = r^2 \gamma' \mu'; \quad K = \frac{S'' + \Sigma_0 + \Sigma_1}{S'}$$

and

$$\Delta_i = \Delta(x_i) = \frac{x_i}{2} \frac{J_0(x_i)}{J_1(x_i)}$$

where J_0 and J_1 are Bessel functions and x_i ($i = 1, 2, 3, \ldots$) are the (real and positive) roots of

$$\Delta(x) - \frac{N_1^2}{R_1 S' \tau} x^2 + K = 0$$

The solutions are:

$$\frac{i_1(t)}{i_{1\infty}} = 1 - \sum_{i=1}^{\infty} \frac{\Delta_i + K}{\Delta_i^2 + x_i^2/4 + K} e^{-\frac{t}{\tau/x_i^2}}; \quad i_{1\infty} = i_1(\infty) = \frac{E_1}{R_1} \tag{80a}$$

$$\frac{\Phi(t)}{\Phi_\infty} = 1 - \sum_{i=1}^{\infty} \frac{1 + K}{\Delta_i^2 + x_i^2/4 + K} e^{-\frac{t}{\tau/x_i^2}}; \quad \Phi_\infty = \Phi(\infty) = \frac{N_1 i_{1\infty}}{S'(1 + K)} \tag{80b}$$

$$\frac{H(t, \varrho)}{H_\infty} = 1 - \sum_{i=1}^{\infty} \frac{(1 + K)\Delta_i}{\Delta_i^2 + x_i^2/4 + K} \frac{J_0\left(\frac{\varrho}{r} x_i\right)}{J_0(x_i)} e^{-\frac{t}{\tau/x_i^2}}; \quad H_\infty = H(\infty, \varrho) = \frac{N_1 i_{1\infty}}{l(1 + K)} \tag{80c}$$

$$\frac{I(t, \varrho)}{I_0} = \sum_{i=1}^{\infty} \frac{\Delta_i + K}{\Delta_i^2 + x_i^2/4 + K} \frac{J_1\left(\frac{\varrho}{r} x_i\right)}{J_1(x_i)} e^{-\frac{t}{\tau/x_i^2}}; \quad I_0 = I(0, r) = \frac{\gamma' E_1}{2\pi r N_1} \tag{80d}$$

b) *Series solutions for small t.*

For small values of t we employ series solutions in powers of $t^{1/2}$, obtained by developing the operators in powers of $p^{-1/2}$. The calculations are similar to those given in Part 3 for a relay *with* magnetic leakage (*cf.* Part 3, eq. (124), for the armature end flux; with leakage the series is in powers of $t^{1/4}$).

The following expressions for the primary current i_1 and the magnetic flux Φ contain terms up to and including t^3 in i_1 and Φ/t:

$$\frac{i_1(t)}{i_{1\infty}} = \frac{2k}{\sqrt{\pi}}\left(\frac{t}{\tau}\right)^{1/2}\left[1+\frac{\sqrt{\pi}A_1}{2}\left(\frac{t}{\tau}\right)^{1/2}+\frac{2A_2}{3}\frac{t}{\tau}+\frac{\sqrt{\pi}A_3}{4}\left(\frac{t}{\tau}\right)^{3/2}+\right.$$

$$\left.+\frac{4A_4}{15}\left(\frac{t}{\tau}\right)^2+\frac{\sqrt{\pi}A_5}{12}\left(\frac{t}{\tau}\right)^{5/2}+\ldots\right]; \quad i_{1\infty}=i_1(\infty)=\frac{E_1}{R_1} \tag{81 a}$$

$$\frac{\Phi(t)}{\Phi_0'} = t\left\{1-\frac{4k}{3\sqrt{\pi}}\left(\frac{t}{\tau}\right)^{1/2}\left[1+\frac{3\sqrt{\pi}A_1}{8}\left(\frac{t}{\tau}\right)^{1/2}+\frac{2A_2}{5}\frac{t}{\tau}+\frac{\sqrt{\pi}A_3}{8}\left(\frac{t}{\tau}\right)^{3/2}+\right.\right.$$

$$\left.\left.+\frac{4A_4}{35}\left(\frac{t}{\tau}\right)^2+\frac{\sqrt{\pi}A_5}{32}\left(\frac{t}{\tau}\right)^{5/2}+\ldots\right]\right\}; \quad \Phi_0'=\left[\frac{d\Phi}{dt}\right]_{t=0}=\frac{E_1}{N_1} \tag{81 b}$$

In these formulæ

$$k=\frac{1}{2}\,\frac{R_1 S'\tau}{N_1^2}$$

and

$$A_1 = -(k-q)$$

$$A_2 = k^2-2qk+3/8$$

$$A_3 = -(k^3-3qk^2+(q^2+3/4)k-3/8)$$

$$A_4 = (k^4-4qk^3+q^2+3/8)k^2-(q+1)3k/4+63/128$$

$$A_5 = -(k^5-5qk^4+(q^2+1/4)6k^3-(q^3+9q/4+9/8)k^2+(q+3/2)3k/4-27/32$$

with

$$q=\frac{1}{2}+2K=\frac{1}{2}+2\,\frac{S''+\Sigma_0+\Sigma_1}{S'}$$

The equivalent winding parameters

Our present, simplified problem is characterized by the conditions

$$S_\lambda=\infty;\ \gamma''=0$$

They give rise to the following numerical values:

$$a = b = \eta^2 = \eta'^2 = \eta''^2 = g = 0$$

$$m_1 = n_1 = p_1 = q_1 = r_1 = t_1 = 1$$

$$\delta' = 1; \quad \delta'' = 0$$

Introducing the total magnetic reluctance

$$\Sigma = \Sigma_0 + \Sigma_1 + S' + S'' = \Sigma_0 + \Sigma_1 + S$$

and observing the relations

$$\lim_{s_\lambda \to \infty} \frac{\eta^2}{g^2 + \eta^2} = \frac{S}{\Sigma}; \quad \lim_{s_\lambda \to \infty} \frac{\beta}{s_\lambda} = \frac{1}{\Sigma_0 + \Sigma_1}$$

we find from (65), (70), (75) or (77), and (79) the following values of the equivalent winding parameters:

$$R_2 = \frac{2\pi}{\gamma' l}; \qquad L_2 = \frac{1}{12 S'} \left(1 + \frac{3S'}{\Sigma}\right) \tag{82 a, b}$$

$$M = \frac{N_1}{2\Sigma}; \quad M_{21} = \frac{1}{2\Sigma} \tag{82 c, d}$$

We add the parameters of the primary winding:

$$L_1 = \frac{N_1^2}{\Sigma}; \quad M_{11} = \frac{N_1}{\Sigma} \tag{82 e, f}$$

The formulæ for primary current and armature end flux (Part 2, eqs. (44) and (46); see also the list of formulæ at the end of this paper) make use of the following quantities:

primary time constant

$$T_1 = \frac{L_1}{R_1} = \frac{N_1^2 / R_1}{\Sigma} \tag{83 a}$$

secondary time constant

$$T_2 = \frac{L_2}{R_2} = \frac{\gamma' l}{24 \pi S'} \left(1 + \frac{3S'}{\Sigma}\right) \tag{83 b}$$

time constant in the formula for the armature end flux

$$T_{21} = T_2 - \frac{M_{21}}{M_{11}} \frac{M}{R_2} = \frac{\gamma' l}{24 \pi S'} \tag{83 c}$$

leakage factor

$$\sigma = 1 - \frac{M^2}{L_1 L_2} = \frac{1}{1 + 3S'/\Sigma} \tag{83 d}$$

Basic data of the quantitative study

The quantitative study of this and the next chapter refers to the relay already investigated in the preceding Parts 1, 2, and 3. The data of this relay were given in Part 3, Chapter 4 (for the geometric data see the second paragraph following the chapter heading; for the electric and magnetic data see Tables 2 and 6).

In short, the data refer to an Ericsson RAB relay with a Lancashire iron yoke and a Lancashire or silicon iron core, an air gap of 0.10 or 0.87 mm, and a winding of 10000 turns with the resistance 560 Ω. In the present chapter we put the leakage reluctances $S_\lambda = \infty$.

Two series of permeability values are employed, *viz.*, the series of high values, corresponding to the mmf 300 At, already used in Part 3, and a series of low values, equal to 1/10 of the values of the first series.

In *Table 1* the different cases treated are specified and suitably designated.

Table 1. The numerical cases treated in Chapters 4 and 5.

Case	Core	Air gap mm	Relative permeability
L87(μ)	Lancashire	0.87	5 400
L10(μ)		0.10	3 400
S87(μ)	Silicon	0.87	4 500
S10(μ)		0.10	3 400
L87(μ/10)	Lancashire	0.87	540
L10(μ/10)		0.10	340
S87(μ/10)	Silicon	0.87	450
S10(μ/10)		0.10	340

Results

Fig. 5 gives primary current and magnetic flux *vs.* time in a case where the eddy current influence stands out clearly. The curves for actual relays with magnetic leakage have the same general appearance.

Without eddy currents both curves are exponential with the time constant 13.7 ms. With eddy currents the flux curve starts with an unchanged rate of increase. As compared to the exponential curve, it then slows down, remaining all the time below this curve. In other words: the eddy currents always cause a decrease of the flux value.

The current curve starts with an infinite rate of increase; we find from the power series development that $di_1/dt \approx \text{const.} \times t^{-1/2}$ for very small values of t. The curve at first rises well above the exponential curve but after a while its rate of increase becomes so small that it cuts across the exponential curve, remaining subsequently below it.

Thus, the primary current curve displays a pronounced knee. In Part 2 we found the same phenomenon in relays with delaying slugs and, in its extreme, in the sleeved relays, where the current rises to its final value almost at once.

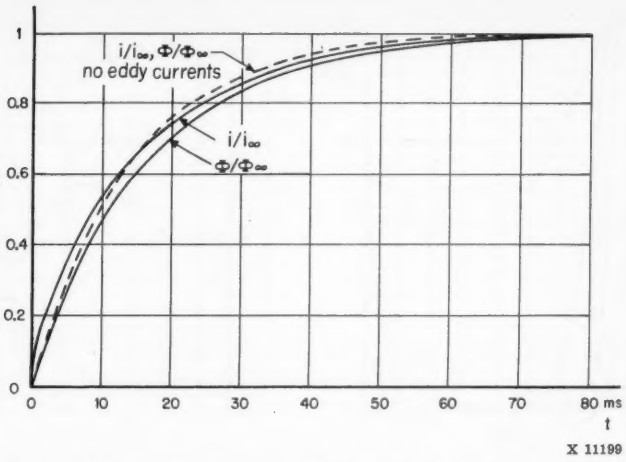

Fig. 5. Primary current and magnetic flux on closing a relay without magnetic leakage and with eddy currents in the core only. Solid curves: case L87(μ/10), Lancashire core, relative permeability of core material = 540, air gap 0.87 mm. Dotted curve: exponential curve, time constant 13.7 ms, valid for relay without eddy currents.

The effects of a change in the permeability value are of interest. An increased permeability of course leads to a larger time constant and hence to an exponential curve with a slower increase. Closer study (*cf.* Figs. 10 a to 13 a) shows that an increase of the permeability entails a decrease of the eddy current influence and a shift toward higher time values of the intersection between the current curve and the exponential curve.

The four pairs of figures, *Figs. 6 a, b to 9 a, b*, give an idea of the accuracy of the equivalent winding solutions. They all refer to the large air gap (0.87 mm), the one of primary interest at closing. The results for the small air gap (0.10 mm) are similar but somewhat more favorable as to the applicability of the equivalent winding.

Figs. 6 a, b refer to case L87(μ), the case with the most marked magnetic skin effect and the largest error in the equivalent winding solution. For four time values, 2.5, 10, 40, and 160 ms, Fig. 6 a gives the distribution across the core section of the magnetic intensity H (to the left), and the eddy current density I (to the right).

One sees that still at 160 ms the distribution of the magnetic flux across the core is extremely inhomogeneous and the eddy current density quite far from the linear increase from core axis to core surface required by our theory (*cf.* eq. (42 a)). Yet, as shown by Fig. 6 b, the equivalent winding solution gives reasonably accurate results except for the primary current at small times.

At the other end we have case S87(μ/10), Figs. 9 a, b, with a fairly small magnetic skin effect. Here the eddy current density follows the linear law much better, and the equivalent winding gives acceptable results down to quite small time values.

Basic data of the quantitative study

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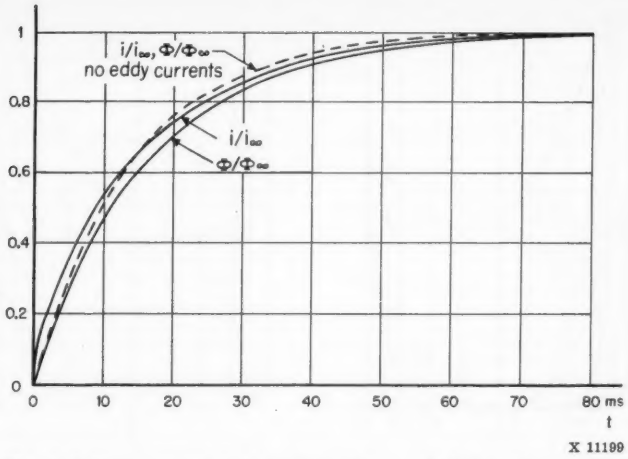

Fig. 5. Primary current and magnetic flux on closing a relay without magnetic leakage and with eddy currents in the core only. Solid curves: case L87(μ/10), Lancashire core, relative permeability of core material = 540, air gap 0.87 mm. Dotted curve: exponential curve, time constant 13.7 ms, valid for relay without eddy currents.

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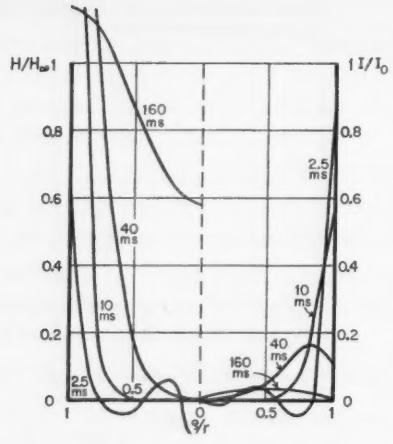
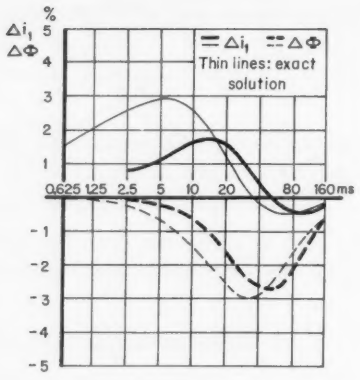


Fig. 6a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case L87(μ).



X 11200

Fig. 6b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only. Case L87(μ).

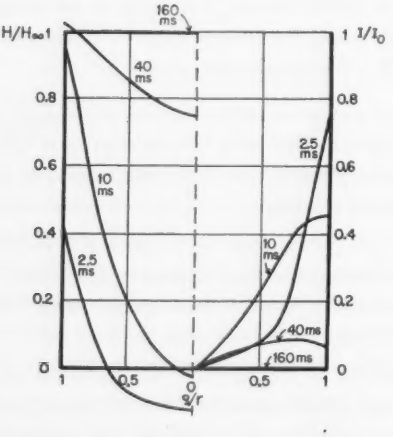
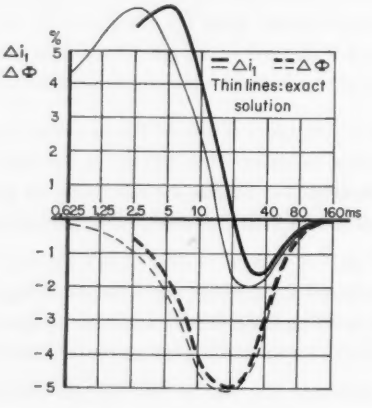


Fig. 7a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case L87(μ/10).



X 11201

Fig. 7b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only. Case L87(μ/10).

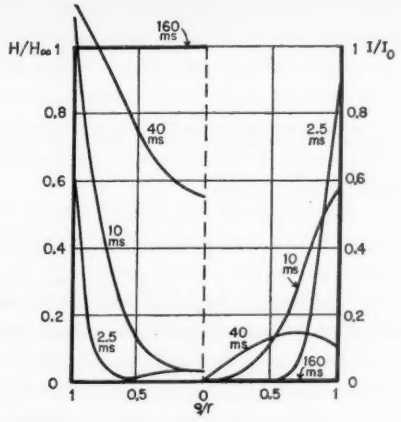
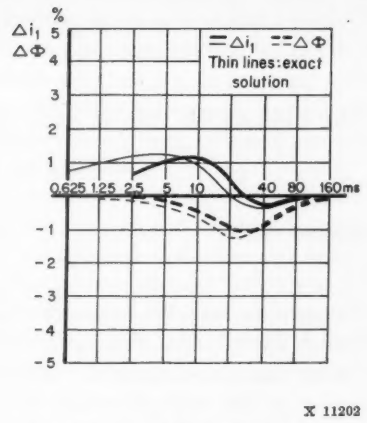


Fig. 8 a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case S87(μ).



X 11202

Fig. 8 b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only. Case S87(μ).

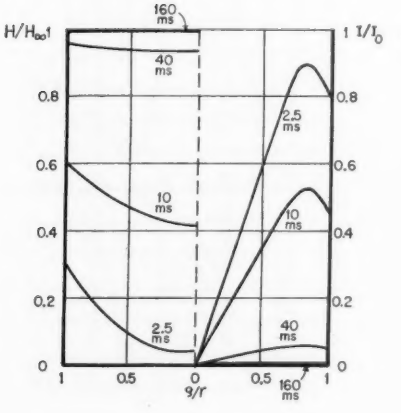


Fig. 9 a. Distribution across the core section of magnetic intensity and eddy current density on closing. No magnetic leakage. Eddy currents in core only. Case S87(μ/10).

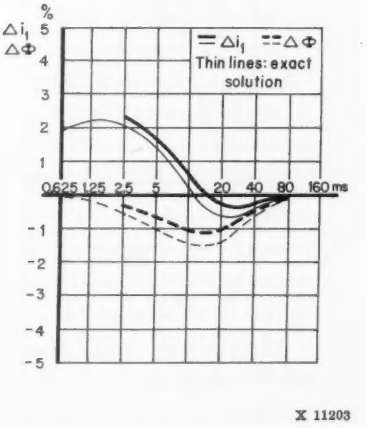


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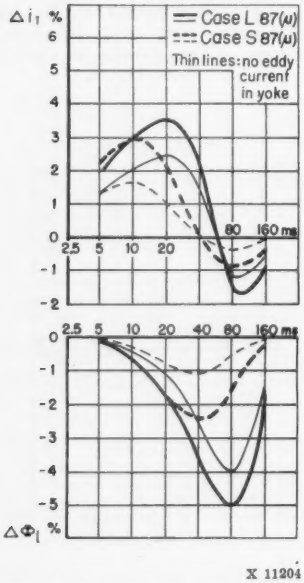
Fig. 9 b. Comparison between the exact and the equivalent winding solutions for primary current and magnetic flux on closing. Deviations from values of relay without eddy currents, in percent of final steady state value. No magnetic leakage. Eddy currents in core only Case S87(μ/10).

CHAPTER 5

The closing of relays with magnetic leakage and with eddy currents in core and yoke

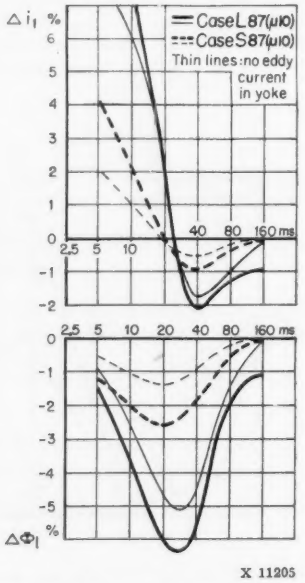
Numerical results

We now turn to the study of actual relays with magnetic leakage and with eddy currents in both core and yoke. We employ the values of the leakage reluctance S_λ, given i Part 3, Chapter 4, Table 2. As in Part 3 (see Part 3, Chapter 4, Table 6 and the two paragraphs above the table), we take account of the eddy currents in the armature and the heel piece, as well as in those parts of the core which are not covered by the winding, by properly increasing the values of the core and yoke conductivities.



X 11204

Fig. 10. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage. Cases L87(μ) and S87(μ).



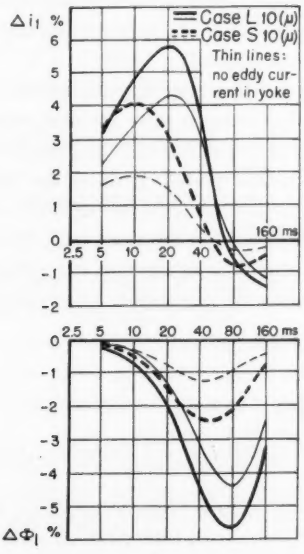
X 11205

Fig. 11. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage. Cases L87(μ/10) and S87(μ/10).

Still, the problem remains so complicated that it is hardly possible to deduce a useful exact solution, that is, to find from the operational expressions given in Part 3 numerically useful time functions. We therefore make use of the equivalent winding. We can do this with great confidence. In fact, for the simplified problem of Chapter 4, the equivalent winding gave us satisfactory results down to at least 5 ms, and neither the magnetic leakage nor the eddy currents in the yoke can be expected to bring about any basic change of our theory.

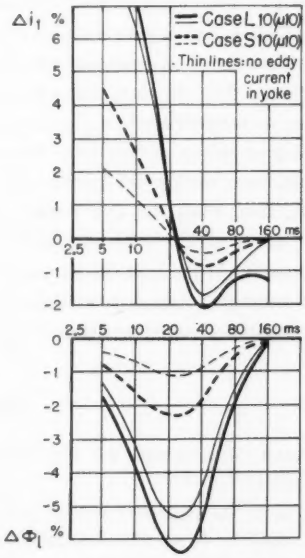
We have calculated, for the eight cases given in Table 1 of the preceding chapter, the equivalent winding parameters, and from these the armature end flux and the primary current on closing. For comparison the cases of eddy currents in the core only and of no eddy currents have also been worked out.

The results are collected in the four *Figs. 10 to 13*. The curves are drawn on a logarithmic time scale. In each case they present the deviation of the actual current or flux value from the corresponding value in a relay without eddy currents. The deviations are given in percent of the final steady state value.



X 11206

Fig. 12. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage.
Cases L10(μ) and S10(μ).



X 11207

Fig. 13. Influence of eddy currents on primary current and armature end magnetic flux on closing. Deviations from values in relay without eddy currents, in percent of final steady state value. Relay with magnetic leakage.
Cases L10(μ/10) and S10(μ/10).

Some general conclusions

The eddy current influence on current and magnetic flux in actual relays on closing is quite small. Therefore, it is not easy to compare our calculations with experimentally determined oscillograph records. From such a comparison it appears, however, that of the two series of permeabilities in Table 1, only the one with high values leads to values of current and flux which are close to the actual ones. By a suitable adjustment of these high values, one could no doubt obtain a very satisfactory agreement with the oscillograms.

In other words, during the closing operation the low initial permeability is of little importance. One should not find this astonishing. To be sure, for small time values the magnetic flux is small but its distribution across the core and yoke sections is strongly inhomogeneous. The flux density and consequently the permeability will therefore reach large values in those parts of the iron which effectively transmit the flux and where the permeability value is hence of importance. One can presume that the permeability value corresponding to the final steady state should yield good results down to fairly small time values.

In our study of the breaking operation in Part 3 the same argument was put forth, the flux on breaking being small but highly inhomogeneous for high time values.

We can therefore state that *both on closing and on breaking, a constant permeability value, in the neighbourhood of the (generally high) value corresponding to the D.C. steady state of the relay, will allow a good representation of the entire course of events.* In both cases the permeability values for small magnetic intensities are of little interest!

It appears that the facts discussed here have not always been duly understood. In the literature on telephone relays a homogeneous flux distribution is often assumed, whereas—on closing as well as on breaking—the magnetic flux in core and yoke is in general strongly inhomogeneous during the major part of the transient period. It should be observed, however, that when the flux passes over to the air gap, it almost immediately assumes a distribution which is very close to the one characterizing the static case. Hence, the inhomogeneity in the iron can be assumed to have only little influence on the pulling force on the armature. Actual force measurements support this conclusion. Its elucidation by further theoretical study would be of value.

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Principal symbols and formulæ

Symbols

t = time.
l = length of core and yoke (length of primary winding).
x = distance along core and yoke, counted from the heel piece end.
r = radius of core.
a = width of yoke.
c = thickness of yoke.

γ' = electric conductivity of core material.
γ'' = „ „ „ yoke „
μ' = magnetic permeability of core material.
μ'' = „ „ „ yoke „ (In Part 3: γ_1, γ_2; μ_1, μ_2.)

N_1 = number of turns of primary winding.
$S' = s'l$ = reluctance of core.
$S'' = s''l$ = „ „ yoke. (In part 3: S_1, S_2.)
$S = S' + S'' = sl$ = reluctance of core + yoke.
Σ_0 = reluctance of heel piece end.
Σ_1 = „ „ armature + air gap.
$S_\lambda = s_\lambda/l$ = total leakage reluctance.
R_1 = resistance of magnetizing (primary) circuit.
L_1 = inductance „ „ „ „
R_2 = resistance of equivalent (secondary) winding.
L_2 = inductance „ „ „ „
M = mutual inductance between primary and equivalent winding.
M_{11} = mutual inductance between primary winding and a fictitious winding of one turn at the armature end.
M_{21} = mutual inductance between equivalent winding and a fictitious winding of one turn at the armature end.

E_1 = constant emf in magnetizing circuit.
i_1 = $i_1(t)$ = current in primary winding.
$i_{1\infty}$ = E_1/R_1 = steady state value of primary current.
i_2 = $i_2(t)$ = current in equivalent winding (total circulating eddy current).
Φ = $\Phi(x, t)$ = total magnetic flux in core and yoke.
$\Phi_l(t)$ = $\Phi(l, t)$ = magnetic flux at the armature end (in Parts 2 and 3: Φ_1).
Φ_∞ = $M_{11}\, E_1/R_1$ = steady state value of armature end flux.
φ_λ = $\varphi_\lambda(x, t)$ = magnetic leakage flux per unit core length.

Formulæ

Auxiliary quantities

$$a=\frac{\Sigma_0}{S_\lambda};\quad b=\frac{\Sigma_1}{S_\lambda} \tag{29a, b}$$

$$\eta=\sqrt{\frac{\overline{S}}{S_\lambda}};\quad \eta'=\sqrt{\frac{\overline{S'}}{S_\lambda}};\quad \eta''=\sqrt{\frac{\overline{S''}}{S_\lambda}} \tag{30a, b, c}$$

$$g^2=\eta^2\,\frac{ab\,\dfrac{\sinh\eta}{\eta}+a\cosh\eta+b}{\eta^2\dfrac{\sinh\eta}{\eta}+b\,(\cosh\eta-1)} \tag{31a}$$

$$\eta^2\ll 1:\quad g^2\approx\frac{a+b+ab+\dfrac{1}{2}\eta^2 a\left(1+\dfrac{1}{3}b\right)}{1+\dfrac{1}{2}b+\dfrac{1}{6}\eta^2\left(1+\dfrac{1}{4}b\right)} \tag{31b}$$

$$m_1=1+a/2-g^2/6 \tag{34a}$$

$$n_1=1+a/3-g^2/12 \tag{34b}$$

$$p_1=1+a/4-g^2/20 \tag{34c}$$

$$q_1=1+a-(g^2-a^2)/3-ag^2/4+g^4/20 \tag{34d}$$

$$r_1=1+a-(g^2-2a^2)/6-ag^2/8 \tag{34e}$$

$$t_1=1+a-(7g^2/4-a^2)/5-ag^2/8+g^4/56 \tag{34f}$$

$$\delta'=\frac{\gamma'/\pi}{\gamma'/\pi+\gamma''c/2a};\quad \delta''=\frac{\gamma''c/2a}{\gamma'/\pi+\gamma''c/2a} \tag{40a, b}$$

$$\beta=\frac{m_1+\dfrac{1}{2}bn_1}{a+b+ab} \tag{59b}$$

Equivalent winding parameters

$$R_2 = \frac{q_1}{m_1^2 l}\left(\frac{2\pi}{\gamma'}\delta'^2 + \frac{16a/3c}{\gamma''}\delta''^2\right) \tag{65}$$

$$L_2 = \frac{1}{12m_1^2 S_\lambda}\left\{\left(1+\frac{\delta''}{3}\right)^2\left(1-\frac{m_1\eta^2}{g^2+\eta^2}\right)\left(3\beta r_1 - \frac{1}{2}t_1\right) + q_1\left(\frac{\delta'^2}{\eta'^2}+\frac{16}{15}\frac{\delta''^2}{\eta''^2}\right)\right\} \tag{70}$$

$$M = \frac{N_1}{2m_1 S_\lambda}\left(1+\frac{\delta''}{3}\right)\frac{q_1}{g^2+\eta^2} \qquad (75),\ cf.\ \text{eq. (77)}$$

$$M_{21} = \frac{1}{2m_1 S_\lambda}\left(1+\frac{\delta''}{3}\right)\left(1-\frac{m_1\eta^2}{g^2+\eta^2}\right)\left[\beta(1+a)-\frac{1}{2}n_1\right] \tag{79}$$

For $S_\lambda = \infty$, $\gamma'' = 0$ (no magnetic leakage, eddy currents in core only), see eqs. (82a—f), (83a—d).

Time constants (cf. Part 2)

$$T_1 = \frac{L_1}{R_1};\quad T_2 = \frac{L_2}{R_2};\quad T_{21} = T_2 - \frac{M_{21}}{M_{11}}\frac{M}{R_2}$$

$$\left.\begin{matrix}\tau_1\\ \tau_2\end{matrix}\right\} = \frac{T_1+T_2}{2} \pm \sqrt{\left(\frac{T_1+T_2}{2}\right)^2 - \sigma T_1 T_2}$$

where

$$\sigma = 1 - \frac{M^2}{L_1 L_2}$$

Primary current and armature end flux (cf. Part 2)

$$\frac{i_1}{i_{1\infty}} = 1 - \frac{(\tau_1 - T_2)e^{-\frac{t}{\tau_1}} - (\tau_2 - T_2)e^{-\frac{t}{\tau_2}}}{\tau_1 - \tau_2};\quad i_{1\infty} = i_1(\infty) = \frac{E_1}{R_1}$$

$$\frac{\Phi_l}{\Phi_\infty} = 1 - \frac{(\tau_1 - T_{21})e^{-\frac{t}{\tau_1}} - (\tau_2 - T_{21})e^{-\frac{t}{\tau_2}}}{\tau_1 - \tau_2};\quad \Phi_\infty = \Phi_l(\infty) = M_{11}\frac{E_1}{R_1}$$

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